MILDLY NONLINEAR FLOW REGIMES OF LIQUID FILMS WITH TWO-AND THREE-DIMENSIONAL WAVES

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It is shown that as the film Reynolds number increases there must be a transition from two-dimensional to three-dimensional waves. All the characteristics of both types of waves are determined.

When the stability of the laminar flow of a thin liquid layer is disturbed (see review in [1]), long two-dimensional waves appear close to the curve of neutral stability; the amplitude of these waves increases monotonically from zero as the supercriticality increases. In natural flows (i.e., without the superimposition of external excitation) a quite specific steady-state wavy regime is set up with characteristics which are uniquely determined by the slope of the support plane and the flow rate and physical properties of the liquid. In contrast to this, the theory usually leads to the conclusion that a single-parametric family of wavy regimes exists, and that in order to discriminate the regime which actually occurs from among the multiplicity of possible regimes it is necessary to make an additional hypothesis.

In this connection, use has been made earlier of the hypothesis of the minimality of the viscous dissipation of energy in the film [2] or of its mean thickness [3] at a given flow rate of the liquid, or of the assumption that for the steady-state, mildly nonlinear, almost harmonic waves, the wavelengths coincide with the analogous value for the wave of maximum growth obtained within the framework of the linear theory [4]. In actual fact there is no need for such additional hypotheses. If it is assumed that a steady-state periodic flow regime with a definite wave length in fact exists and is stable, then it must possess the following properties. Firstly, the corresponding increment of growth of the fluctuation (the imaginary part of the complex wave frequency) must tend to zero (the condition for the steady state to exist). Secondly, the waves whose lengths differ from those of the favored waves must be damped, i.e., the zero value of the increment, considered as a function of the wave number with all the other parameters constant, must be a maximum (the periodicity condition). These conditions, which were first introduced for one-dimensional flows in [5], appear to be sufficient to uniquely define the characteristics of the steady-state, mildly nonlinear regime of flow of liquid films with two-dimensional waves [6, 7].

As shown below, analogous arguments make it possible to determine the characteristics of steady-state regimes with three-dimensnional waves, and also to determine the conditions under which the transition occurs from two-dimensional to three-dimensional flow. Bearing in mind mainly the principles of the matter, we will use here only the simplest evolution equations for the relative liquid film thickness:

$$\frac{\partial\varphi}{\partial t}+3(1+\varphi)^2\frac{\partial\varphi}{\partial x}+\frac{6}{5}\operatorname{Re}\frac{\partial^2\varphi}{\partial x^2}-(\operatorname{tg}\alpha)\Delta\varphi+\frac{\mathrm{We}}{\cos\alpha}\Delta^2\varphi=0, \tag{1}$$

$$\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

Here the following parameters have been introduced:

$$\varphi=\frac{h-h_0}{h_0},\ \operatorname{Re}=\frac{u_0h_0}{\nu},\ \mathrm{We}=\frac{\sigma}{\rho g h_0^2},$$

$$u_0=\left(\frac{\cos\alpha}{3}\frac{g}{\nu}\right)^{1/3}Q^{2/3},\ h_0=\left(\frac{3}{\cos\alpha}\frac{\nu Q}{g}\right)^{1/3}. \tag{2}$$

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The x axis is directed downwardly with respect to the flow, with the y axis normal to it in the plane of the support; the z axis is normal to this plane. The dimensional coordinates x y, z are referred to the scale parameter h_0.

The dimensionless velocity component in the direction of the flow referred to the scale u_0 can be represented in the following form to an accuracy corresponding to that assumed in (1):

$$v_x = 3\left[1-(\operatorname{tg}\alpha)\frac{\partial\varphi}{\partial x}+\frac{\mathrm{We}}{\cos\alpha}\frac{\partial}{\partial x}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)\right]\left(1+\varphi-\frac{z}{2}\right)z. \tag{3}$$

Equation (1) and the expression (3) are valid for three-dimensional flow when $\varphi \ll 1$, $\varepsilon\mathrm{Re} \leqslant \varphi^2$, $\operatorname{tg}\alpha \leqslant 1$, $\varepsilon^2\mathrm{We} \leqslant 1$ ($\varepsilon = h_0/\lambda$, where λ is the wave length), and are obtained from the system of equations for the hydrodynamics to the thin liquid film approximation and the corresponding boundary conditions completely analogously to the evolution equation and the expression for v_x in two-dimensional flows, which are valid for the same conditions [6]. In fact, the latter are very restrictive, and we should consider instead of (1) a more complex variant of the evolution equation whose analogy for two-dimensional flow is formulated in [6, 7]. The use of (1) has the advantage that all the important results can be obtained analytically without the need to fall back on numerical investigations.

Let us represent the unknown Eq. (1) in the form (*):

$$\varphi(x, y, t) = \sum_{n=-\infty}^{\infty} \varphi_n(y)\, e^{in(\omega t - kx)}, \quad \varphi_n(y) = \Phi_n(1+\psi_n(y)),$$

$$\psi_n(y) = \sum_{m=-\infty,\ m\neq 0}^{\infty} \Psi_{nm} e^{inmly}, \quad \varphi_{-n}(y) = \varphi_n^*(y) \tag{4}$$

(the superscript asterisk denotes complex conjugation), where the wave numbers k and l are assumed to be real and nonnegative. Because of the choice of the origin from which the time is meaasured or the longitudinal coordinate, it is possible to make the amplitude of the main harmonic Φ_1 real and positive; it is assumed furthermore that $\Phi_1 = \sqrt{q}$, $q \ll 1$. From a simple analysis, it follows that $|\Phi_n| \sim q^{|n|/2}$, $|\Phi_0| \sim q$ [5] . Equation (1) was derived to an accuracy of terms of the order $|\varphi|^3 \sim q^{3/2}$ inclusive. Hence, in the use of q as a small parameter in the sum with respect to n in Eq. (4) it is necessary to retain only the harmonics with $|n| \leq 2$ in avoiding exceeding the accuracy. In the sums with respect to m it is also necessary to take into account only the terms with $|m| \leq 2$, since it can be shown that the retention of terms with $|m| > 2$ leads to the appearance in the ratio obtained previously of quantities whose order is not less than q^2.

By substituting (4) into (1) and collecting the terms proportional to the various harmonics in the usual way, a system of algebraic equations is obtained for the coefficients Φ_n and Ψ_{nm} introduced in Eq. (4). An additional equation (which makes it possible to determine Φ_0) can be found from the condition that the flow rate of the liquid, averaged with respect to x and y, where the liquid flows with the velocity (3), can be equated to the corresponding quantities in a plane-parallel flow (i.e., to unity in terms of the dimensionless variables which have been introduced). From these equations, the following relationships are obtained after simple though cumbersome computations

$$\Phi_0 = -2(1+2\Psi_{11}\Psi_{-11})q, \quad \Phi_2 = \frac{3ik}{a_3}(1+2\Psi_{11}^2)q,$$

*Strictly speaking, the function (4) can represent the solution of the problem being considered only if ω is also real. If $\omega = \Omega - i\gamma$, and $\gamma \neq 0$, then (4) is not a solution. At the same time, to obtain a positive value for the periodicity, as mentioned above, it is necessary to consider the solution not only at the point $\gamma = 0$ (which is quite sufficient for deriving the relationship for the characteristics of steady-state waves from their lengths and physical and regime parameters), but also at other points, even though these may be only a small distance from this point. The latter makes the computations quite cumbersome. Hence, in the text the evaluation is carried out using the expansion (4) formally and with $\gamma \neq 0$, while the basis for the validity of this procedure and the condition of periodicity, which makes it possible to determine the length of the steady-state waves in practice, are given in the Appendix.

$$\Phi_2\Psi_{21}=\frac{6ik}{a_4}\Psi_{11}q,\quad \Phi_2\Psi_{22}=\frac{3ik}{a_5}\Psi_{11}^2q,$$
$$\Psi_{12}=-\frac{3k}{a_2}\left[\left(\frac{6k}{a_5}-i\right)\Psi_{11}^2+2\left(\frac{6k}{a_4}-i\right)\Psi_{11}\Psi_{-11}\right]q. \tag{5}$$

In addition, by substituting (5) into the remaining unused algebraic equations for the coefficients, it is found that:

$$a_0+3k\left\{\frac{6k}{a_3}+3i+\left[2(v+iw)\left(\frac{6k}{a_3}-i\right)+\frac{24k}{a_4}+4i\right]r\right\}q=0,$$
$$a_1+3k\left\{\frac{12k}{a_4}+2i+(v-iw)\left(\frac{6k}{a_3}-i\right)+\left(\frac{12k}{a_3}+\frac{6k}{a_5}+5i\right)r\right\}q=0. \tag{6}$$

Here we have introduced the variables

$$r=\Psi_{11}\Psi_{-11}=|\Psi_{11}|^2,\quad v^2+w^2\equiv 1,$$
$$v=\frac{1}{2}\left(\frac{\Psi_{11}}{\Psi_{-11}}+\frac{\Psi_{-11}}{\Psi_{11}}\right),\quad w=-\frac{i}{2}\left(\frac{\Psi_{11}}{\Psi_{-11}}-\frac{\Psi_{-11}}{\Psi_{11}}\right) \tag{7}$$

and the notation

$$a_0=i(\omega-3k)-Ak^2+Bk^4,\quad a_1=i(\omega-3k)-Ak^2+B(k^2+l^2)^2+(\operatorname{tg}\alpha)l^2,$$
$$a_2=i(\omega-3k)-Ak^2+B(k^2+4l^2)^2+4(\operatorname{tg}\alpha)l^2,\quad a_3=i(\omega-3k)-2Ak^2+8Bk^4,$$
$$a_4=i(\omega-3k)-2Ak^2+8B(k^2+l^2)^2+2(\operatorname{tg}\alpha)l^2, \tag{8}$$
$$a_5=i(\omega-3k)-2Ak^2+8B(k^2+4l^2)^2+8(\operatorname{tg}\alpha)l^2,$$
$$A=\frac{6}{5}\mathrm{Re}-\operatorname{tg}\alpha,\quad B=\frac{\mathrm{We}}{\cos\alpha}.$$

From the requirement of compatibility of the equations in (6) when $r = 0$, $l = 0$, it follows that $v = -1$, $w = 0$; when Eq. (7) is taken into account it is therefore found that $v = -\sqrt{1-w^2}$.

For given values of k, l, and q, Eq. (6) serves for determining the complex quantities Ψ_{11} and $\omega = \Omega - i\gamma$; the remaining coefficients from (4) can be found in accordance with (5). In addition to (6), according to what was noted above, for steady-state, almost harmonic waves there also exists the equation:

$$\gamma=0,\quad \partial\gamma/\partial k=0,\quad \partial\gamma/\partial l=0, \tag{9}$$

from which it is possible in principle to determine both the wave numbers and the amplitudes of the main harmonics. Thus, the four effective equations resulting from (6) and the equations (9) make up a system of seven equations for determining the real quantities γ, Ω, k, l, and q, and also the modulus and argument of the complex coefficient Ψ_{11}. The steady-state wavy regime of the type being considered is possible if the extremal value of γ actually occurs in the open first quadrant of the (k, l) plane and represents a maximum of γ, with the corresponding $q > 0$. A maximum of γ can also be reached in principle on the boundary $l = 0$ of this quadrant; in this case, the regime of flow with two-dimensional waves is possible.

Let us now give the results of an investigation of this system of algebraic equations. First, when $q \to 0$ it is found from (6) that $a_0 = a_1 = 0$, which represents the known dispersion relationship of the linear theory for two- and three-dimensional waves. The stability of the laminar plane-parallel flow is disrupted when

$$\mathrm{Re}>\mathrm{Re}_{*1}=5/6\operatorname{tg}\alpha \tag{10}$$

with respect to waves with $k = l = 0$, while (9) determines the wave number of the wave of maximum growth when $Re > Re_{*1}$. The critical value Re_{*1} of the film Reynolds number determines the first bifurcation of the evolution equation (1).

If $q \neq 0$, and Re only slightly exceeds Re_{*1}, then the maximum is achieved at $l = 0$, i.e., for two-dimensional waves. In this case, the system of equations reduces to the form considered in [6]:

$$k = k_0 = \left(\frac{8+\sqrt{43}}{21}\,\frac{A}{B}\right)^{1/2} \approx 0{,}833\left(\frac{A}{B}\right)^{1/2},\quad q = q_0 \approx 0{,}046\,\frac{A^3}{B},$$
$$\Omega = \Omega_0 = c_0 k_0,\quad c_0 = 3(1 - 3q_0) \approx 3(1 - 0{,}138A^3/B), \tag{11}$$

where A can be represented as $6/5(Re - Re_{*1})$. The last of the equations of Eq. (9) was not considered in obtaining (11).

In the general case, the system of seven equations can be conveniently solved as follows. By expressing the quantities Ω and γ from both of the equations in (6) and equating the representations which are obtained, it is found that

$$w = -(k/6)(7Bk^2 - A)\,r, \tag{12}$$

$$\Omega = 3\left[(1-3q)\,k - \left(\frac{12}{7Bk^2 - A}\,w + 2k\sqrt{1-w^2} + 4k\right) rq\right], \tag{13}$$

and also, by taking into account (8) and (12):

$$\begin{aligned}\gamma = {} & Ak^2 - Bk^4 - 18(7Bk^2 - A)^{-1}q + 36\{(7Bk^2 - A)^{-1} - \\ & - 2k^2[8B(k^2 + l^2)^2 - Ak^2 - Bk^4 + 2(\operatorname{tg}\alpha)\,l^2]^{-1}\}\,rq = \\ = {} & Ak^2 - B(k^2 + l^2)^2 - (\operatorname{tg}\alpha)\,l^2 + 18\{(7Bk^2 - A)^{-1} - \\ & - 2k^2[7B(k^2 + l^2)^2 - Ak^2 + (\operatorname{tg}\alpha)\,l^2]^{-1}\}\,q + k^2\{1/2\,(7Bk^2 - A) - \\ & - 36[8Bk^4 - Ak^2 - B(k^2 + l^2)^2 - (\operatorname{tg}\alpha)\,l^2]^{-1} - \\ & - 18[B(7k^4 + 62k^2l^2 + 127l^4) - Ak^2 + 7(\operatorname{tg}\alpha)\,l^2]^{-1}\}\,rq.\end{aligned} \tag{14}$$

Thus, from (9) and (14) there is a system of four equations for determining k, l, q, and r. It is assumed that the regime of flow with three-dimensional waves is possible when $Re > Re_{*2}$, and this system of equations is solved by the method of small parameters, using expansions of the unknowns with respect to powers of β, i.e.,

$$k^2 = k_0^2 + \beta k_1^2 + \ldots,\quad q = q_0 + \beta q_1 + \ldots,$$
$$l^2 = \beta l_1^2 + \ldots,\quad r = \beta r_1 + \ldots,\quad \beta = Re - Re_{*2}. \tag{15}$$

Hence, it is not difficult to obtain equations corresponding to different approximations with respect to β. From the equation of the zeroth approximation it is possible to find firstly equation (11) for k_0 and q_0, and secondly, the equation

$$Re_{*2} \approx 5{,}23\,\operatorname{tg}\alpha, \tag{16}$$

which was obtained from the last equation of Eq. (9). This formula determines the second bifurcation corresponding to the transition from the steady-state two-dimensional to the steady-state three-dimensional wavy-flow regime.

The equations of the first approximation give

$$k_1^2 \approx \frac{0{,}832}{B},\quad l_1^2 \approx \frac{0{,}017}{B},\quad q_1 \approx 0{,}164\,\frac{A_*^2}{B},\quad r_1 = 0,$$
$$A_* = 6/5Re_{*2} - \operatorname{tg}\alpha \approx 5{,}28\,\operatorname{tg}\alpha. \tag{17}$$

From (12) and (13) it is now easy to obtain corresponding to (15) representations for w, Ω, and also for the phase velocity c of the traveling waves. Thus,

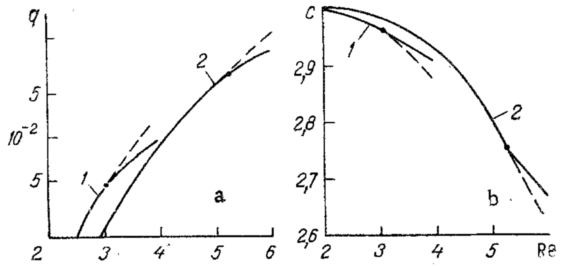

Fig. 1. Dependence of the square of the wave amplitude (a) and of the phase velocity (b) on the Reynolds number for $\alpha = \pi/6$ and $\pi/4$ (curves 1 and 2, respectively); the points correspond to the transition from two- to three-dimensional waves; the dashed lines refer to two-dimensional waves in the zone where three-dimensional waves occur.

$$w = \beta w_1 + \ldots,\ \Omega = \Omega_0 + \beta\Omega_1 + \ldots,\ c = c_0 + \beta c_1 + \ldots, \tag{18}$$

while for Ω_0 and c_0 the formulas in (11) are valid as before, and

$$\Omega_1 = \frac{3}{2}\frac{k_1^2}{k_0}(1 - 3q_0) - 9k_0q_1,\ \ w_1 = 0,\ c_1 = -9q_1. \tag{19}$$

By making use of (5) it is possible to find an expansion of the coefficients in the sums of (4) in terms of powers of the supercriticality β, and to write a final expression for $\varphi(x, y, t)$. In order to conserve on space, these expressions are not given, but it can be noted that in spite of the equality of the quantity r_1 to zero (and consequently, the absence of a term proportional to β in the expansion for Ψ_{11}), the coefficients for β in the expansions of Ψ_{12} and Ψ_{21} appear to be different to zero. In order to determine the main terms $\beta^2 r_2$ and $\beta^2\omega_2$ in the expansions of r and ω it is necessary to consider the system of equations to the second approximation. The result is

$$r_2 \approx \frac{0{,}0007}{1 + 0{,}095A_*^3/B}\frac{A_*}{B},\ w_2 = -\frac{k_0(7Bk_0^2 - A_*)}{6}r_2. \tag{20}$$

It can be shown that the increment γ reaches its maximum precisely at the point being considered in the plane (k, l).

Thus, the parametric plane (Re, α) can be brokwn down into three zones. The zone Re < Re_{*1} corresponds to stable plane-parallel flow, the zone Re_{*2} > Re > Re_{*1} to steady-state two-dimensional wavy flow, and the zone Re > Re_{*2} to steady-state flow with three-dimensional waves. For films on a vertical wall both critical values of the Reynolds number tend towards zero, i.e., only the third zone exists. The dependence of q and c on Re at $\alpha = \pi/6$ and $\pi/4$ (curves 1 and 2, respectively) are shown in Fig. 1. The solid lines correspond to the regime which is actually established, while the dashed lines correspond to the regime with two-dimensional waves in the zone Re > Re_{*2}; the points mark the bifurcations. For the three-dimensional waves the amplitude q increases and the velocity c decreases with increase of Re in the zone Re > Re_{*2} somewhat more slowly than for the two-dimensional flows. On the whole, this is in agreement with the experimentally observed trends [1].

Equation (1) is approximately valid at small values of the Reynolds number [6]. It can be extended into the zone of larger Reynolds numbers by using other, more complex, evolution equations, analogously to the equations for two-dimensional wavy flows in [6, 7]. It is not difficult to obtain such equations, but the calculation of the parameters of the three-dimensional wavy regime appears to be very time-consuming. The results [7] obtained for two-dimensional wavy flow can be regarded as approximations to the real three-dimensional flows which are established at larger values of Re.

It should be noted that in order to obtain physically comprehensible results when Re $\gtrsim$ 1 it is not sufficient to simply increase the number of the terms taken into account in the expansion (4) while leaving the evolution equation unchanged. As noted previously in [6], this would mean an increase in the accuracy of the evolution equation. In general, attempts to construct waves which are strongly nonlinear (when $\varphi \sim 1$) on the basis of an analysis of equations obtained with the assumption that $\varphi << 1$ are physically absurd in the opinion of the authors, and at best can be regarded as purely mathematical exercises. Examples of such attempts can be found in [8], and also in a number of other papers.

Investigations of slightly nonlinear three-dimensional waves are known from the literature [9-11]. However, as in the case of many analogous papers on two-dimensional waves also, the wave lengths in the longitudinal and transverse directions or the corresponding wave numbers are regarded as quantities which are specified a priori. In this connection, it is possible to note the difference of the present work from these investigations; the main result of the present work consist not so much of determining the dependence of the characteristics of the wavy regime on these wave numbers as indicating the route to be used in principle for determining the latter.

In conclusion it should be noted that here we have not considered the problem of the stability of the steady-state regimes with two- and three-dimensional waves which have been considered. The effective solution of this problem constitutes an independent topic and can obviously be obtained by using the method proposed in [12] and further developed in a number of subsequent papers.

APPENDIX

We will confine our attention to the analysis of the equation:

$$\frac{\partial f}{\partial t}+F\left(f,\ \frac{\partial f}{\partial x},\ \frac{\partial^2 f}{\partial x^2},\ \ldots\right)=0 \tag{A1}$$

with a single spatial variable x, in which F is some nonlinear function (it may in addition depend on the parameter). Generalization to the situation with two spatial parameters does not present difficulties in principle, but complicates the computations. Let us assume that (A1) has a single periodic solution corresponding to a stationary wave, which can be represented in the form of the convergent series

$$f_*(t,\ x)=\sum_{n=-\infty}^{\infty}\overset{\circ}{\Phi}_n\, e^{in(\Omega_* t-k_* x)},\quad \overset{\circ}{\Phi}_{-n}=\overset{\circ}{\Phi}{}_n^{*}. \tag{A2}$$

The existence and uniqueness of the solution (A2) means that the values of the constants $\overset{\circ}{\Phi}_n$, Ω_*, k_* are uniquely determined.

Let us consider the nonsteady-state solution of Eq. (A1):

$$f_k(t,\ x)=\sum_{n=-\infty}^{\infty}\Phi_n(t)\, e^{in(\Omega t-kx)},\quad \Phi_{-n}(t)=\Phi_n^*(t), \tag{A3}$$

where Ω, k are real positive numbers. By substituting (A3) into (A1) a system of equations is obtained after simple rearrangements for the coefficients of the series (A3):

$$\frac{d\Phi_n}{dt}=\alpha_n\Phi_n+\sum_{l=-\infty}^{\infty}\alpha_{nl}\Phi_{n-l}\Phi_l+\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\alpha_{nlm}\Phi_{n-l}\Phi_{l-m}\Phi_m+\ldots, \tag{A4}$$

where the values of α, which depend on Ω and k, are completely determined by the form of the function F.

Feebly nonlinear processes described by the solutions (A2) or (A3) usually develop from mild disturbances of the stability of some steady-state process which is described by a solution of Eq. (A1) which does not depend on t. It is known that in this case the amplitudes of the various harmonics in (A2) and (A3) satisfy the sequence relationship

$$|\Phi_0|\sim q,\ |\Phi_n|\sim q^{|n|/2},\ n\neq 0,\ q=|\Phi_1|^2$$

and can be represented in the form of series

$$\Phi_0=\Phi_1^2\sum_{j=0}^{\infty}\beta_{0j}q^j\ ,\ \Phi_n=\Phi_1^{|n|}\sum_{j=0}^{\infty}\beta_{nj}q^j\ ,\ n\neq 0, \tag{A5}$$

which converge at sufficiently small values of q and are termwise differentiable with respect to the variable t, which in this case plays the role of a parameter (see, for instance, [5], and also [13, 14]). The coefficients β depend on k and Ω.

It is clear that it is sufficient to consider Eq. (A4) only for the case when $n \geq 0$. The right-hand part of any such equation can be expressed directly by using (A5) in the form of an expansion in powers of q multiplied by $\Phi_1{}^n$. An analogous representation for the left-hand side is obtained after differentiation of the corresponding series (A5), substitution into the result of the expressions for $d\Phi_1/dt$ and $d\Phi_1^*/dt = d\Phi_{-1}/dt$ which result from (A4) when $n = \pm 1$, and using the series (A5). By equating these expansions, equations are obtained which confirm the equality to zero of the products of $\Phi_1{}^n$ with an infinite sum of terms proportional to $q^m (m \geq 0)$. Since in this case q must be regarded as an arbitrary quantity, the coefficients on the various powers of q must be separately equal to zero. A system of recursive equations follows from this from which it is not difficult to find in turn all the coefficients β which appear in (A5) as functions of the quantities α introduced in (A4). Because of their extremely cumbersome form, all these equations are not written out, but the expression

$$\beta_{20} = \alpha_{21} (2\alpha_1 - \alpha_2)^{-1} \tag{A6}$$

is given for the coefficients determining the main terms in the expansion (A5) for the amplitude of the second harmonic.

By substituting (A5) into Eq. (A4) with n = 1, it is found that

$$\frac{d\Phi_1}{dt} = \Phi_1 H, \; H(\Omega, \; k, \; q) = \sum_{j=0}^{\infty} \eta_j q^j \; , \; q = |\Phi_1|^2, \tag{A7}$$

where η is described in a specified way in terms of α and β, i.e., η also depends on k and Ω. By multiplying (A7) by Φ_1^* and combining the equations which are obtained with the complex conjugates, it is found that

$$\frac{dq}{dt} = 2q\Gamma, \; \Gamma(\Omega, \; k, \; q) = \sum_{j=0}^{\infty} \gamma_j q^j \; , \; \gamma_j = \operatorname{Re} \eta_j. \tag{A8}$$

The function Γ in the right-hand part of (A8) has the significance of a nonlinear growth increment of the main (first) harmonic. The periodic solution $\Phi_1 = \overset{\circ}{\Phi}_1$, $q = q_* = |\overset{\circ}{\Phi}_1|^2$ corresponds to the solution (A2); in addition, the right-hand parts of equations (A7) and (A8) should tend to zero, which determines the parameters k_* and Ω_*. The remaining coefficients $\overset{\circ}{\Phi}_n$ in (A2) are evaluated from (A5) with the given values of the parameters. (It should be noted that in fact all such evaluations are carried out with some precision with respect to small values of q_* which is specified a priori.)

A function $\gamma(k) = \Gamma(\Omega(k, q_*), k, q_*)$ is now introduced, where $\Omega(k, q_*)$ is the solution to the equation Im H = 0 with $q = q_*$. The "condition of steady-state behavior" introduced at the beginning of the paper in this way represents a consequence of the assumed existence of the periodic solution (A2) of the equation (A1). Let us now show that the "condition of periodic behavior" $\partial\gamma(k_*)/\partial k_* = 0$ represents a consequence of the assumed uniqueness of this periodic solution.

Let us assume that the value of $\gamma(k)$ is negative when $k \to 0$ and $k \to \infty$, and assume that on the contrary, $\gamma(k_*) = 0$, but $\partial\gamma(k_*)/\partial k_* \neq 0$. Then such a value k_{**} is found, not equal to k_*, that $\gamma(k_{**}) = 0$. Assuming $\Omega_{**} = \Omega(k_{**}, q_*)$, $\Phi_1' = \overset{\circ}{\Phi}_1$, and evaluating the remaining values of $\Phi_n' = \Phi_n$ from (A5) with the given values of k_{**} and Ω_{**}, it is seen that the series

$$f_{**}(t, \; x) = \sum_{n=-\infty}^{\infty} \Phi_n' e^{in(\Omega_{**}t - k_{**}x)}, \; \Phi_n' \neq \overset{\circ}{\Phi}_n, \; n \neq \pm 1,$$

will also be a periodic solution of (A1) which is different to (A2), which contradicts the assumption of the uniqueness of this solution.

The equations $\gamma(k_*) = 0$ and $\partial\gamma/\partial k_* = 0$ determine (for known $\Omega(k_*, q_*)$) both k_* and also the quantity q_*, which up to now have been considered as parameters. In the discussion given above the important assumption of the negative nature of $\gamma(k)$ when $k \to 0$ and $k \to \infty$ must be confirmed a posteriori.

Let us now show that the function $\gamma(k)$ which has been introduced coincides with the analogous function which must be set up formally using the series (4) with $\gamma \neq 0$. For simplicity, let us again consider the one-dimensional problem (plane waves), when Eq. (1) coincides with the equation already investigated in [6], i.e.,

$$\frac{\partial\varphi}{\partial t} + 3(1+\varphi)^2\frac{\partial\varphi}{\partial x} + A\frac{\partial^2\varphi}{\partial x^2} + B\frac{\partial^4\varphi}{\partial x^4} = 0,$$
$$A = \frac{6}{5}\,\mathrm{Re} - \mathrm{tg}\,\alpha,\ B = \frac{\mathrm{We}}{\cos\alpha}. \tag{A9}$$

In the calculations, only the zeroth and second harmonics will be taken into account in addition to the first harmonic, i.e., it is assumed that

$$\varphi_k(t,\ x) = \sum_{n=-2}^{2}\Phi_n(t)\,e^{in(\Omega t - kx)},\ \Phi_{-n}(t) = \Phi_n^*(t). \tag{A10}$$

By substituting (A10) into (A9), a system of equations of the type (A4) is obtained for the problem being investigated:

$$\frac{d\Phi_1}{dt} = -[i(\Omega - 3k) - Ak^2 + Bk^4]\,\Phi_1 + 3ik\Phi_{-1}\Phi_1^2 + 6ik(\Phi_2\Phi_{-1} + \Phi_0\Phi_1),\ d\Phi_0/\partial t = 0,$$
$$\frac{d\Phi_2}{dt} = -[2i(\Omega - 3k) - 4Ak^2 + 16Bk^4]\,\Phi_2 + 6ik\Phi_1^2. \tag{A11}$$

With $n = 0$, the equation appears to be degenerate. The constant Φ_0 is determined from the condition for the equality of the nondimensional flow rates in wavy film flow and in plane-parallel film flow [6], whence $\Phi_0 = -2\Phi_1^\circ\Phi_{-1}^\circ$. The solution of the third equation of (A11) becomes, in accordance with (A6):

$$\Phi_2 = 3i\,[k(7Bk^2 - A)]^{-1}\Phi_1^2.$$

In this case, Eq. (A8) has the form

$$\frac{dq}{dt} = 2q\left(Ak^2 - Bk^4 - \frac{18}{7Bk^2 - A}\,q\right),$$

and hence,

$$\gamma(k) = Ak^2 - Bk^4 - \frac{18}{7Bk^2 - A}\,q_*,$$

which in accuracy agrees with the corresponding expression for $\gamma(k)$ obtained in [6].

NOTATION

A, a_n, B, quantities introduced in (8); c, dimensionless phase velocity; g, acceleration of gravity; h, h_0, film thicknesses in the wavy and nonwavy regimes; k, l, wave numbers; Q, liquid flow rate; q, square of the amplitude of the main harmonic; r, quantity introduced in (7); t, dimensionless time; u_0, mean velocity in nonwavy regime; v, w, quantities introduced in (7); v_x, longitudinal component of dimensionless velocity; x, y, z, dimensionless coordinates; α, angle between the plane of the substrate and the vertical; β, supercriticality parameter; γ, growth increment of perturbation; ε, long wavelength parameter; λ, linear longitudinal scale; ν, kinematic viscosity; ρ, liquid density; σ, surface tension coefficient; Φ_n, φ_n, ψ_n, Ψ_{nm}, functions in (4); φ, dimensionless wave amplitude; Ω, ω, real and complex frequencies; Re, We, Reynolds and Weber numbers; Re_{*_1}, Re_{*_2}, critical values of the Reynolds number; *, superscript asterisk denotes complex conjugate.

LITERATURE CITED

1. V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, The Propagation of Waves in Gas- and Vapor—Liquid Media [in Russian], Novosibirsk (1983).

2. P. L. Kapitsa, Zh. Éksp. Teor. Fiz., 18, No. 1, 3-28 (1948).
3. V. Ya. Shkadov, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 1, 43-51 (1967).
4. S. V. Alekseenko, B. G. Pokusaev, and V. E. Nakoryakov, Prikl. Mekh. Tekh. Fiz., No. 6, 77-87 (1979).
5. Yu. B. Ponomarenko, Prikl. Mat. Mekh., 28, No. 4, 688-698 (1964).
6. Yu. A. Buevich and S. V. Kudymov, Prikl. Mekh. Tekh. Fiz., No. 1, 15-20 (1983).
7. Yu. A. Buevich and S. V. Kudymov, Inzh.-Fiz. Zh., 45, No. 4, 566-576 (1983).
8. O. Yu. Tsvelodub, Wave Processes in Two-Phase Media [in Russian], Novosibirsk, (1980), pp. 47-63.
9. A. A. Nepomnyashchii, Hydrodynamics [in Russian], No. VII, 43-52 (1974).
10. E. A. Demekhin and V. Ya. Shkadov, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5, 21-27 (1984).
11. Yu. Ya. Trifonov and O. Yu. Tsvelodub, Hydrodynamics and Heat and Mass Transfer of Liquid Flows with Free Surfaces [in Russian], Novosibirsk (1985), pp. 85-102.
12. A. A. Nepomnyashchii, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3, 28-33 (1974).
13. L. D. Landau and E. M. Lifshits, Hydrodynamics [in Russian], Moscow (1986).
14. Yu. B. Ponomarenko, Prikl. Mat. Mekh., 32, No. 1, 46-58 (1968).

FLOW OF A NON-NEWTONIAN LIQUID IN THE GAP BETWEEN A ROTATING CYLINDER AND A PERMEABLE SURFACE WITH ROTOR GRANULATION

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The isothermal process of rotor granulation of a material having the properties of an anomalously viscous liquid is analyzed hydrodynamically.

One of the highly promising methods for processing of highly viscous media is rotor granulation. Rotor machines which combine the functions of a pump and a forming device are characterized by minimum deformation of the material being processed and permit granulation of highly filled heterogeneous systems. Rotor-type granulators are widely used for processing of pastelike materials, suspensions, and polymers in the pharmaceutical, food, and metallurgical industries, for production of plastics and rubber parts, in mineral fractionation, and a number of other chemical technology processes [1].

The available theoretical studies of material flow in granulators [1-3] contain inaccuracies in formulation of the boundary problem. Thus, for example, their authors assume that flow terminates in a minimal gap and that excess pressure is equal to zero. This corresponds to the Ardichvili concept for a roller process in which the flow occurs at zero matrix permeability [4].

The present study will attempt a hydrodynamic analysis of flow of a non-Newtonian (power-law) liquid in a rotor granulator corresponding to the Gaskell concept for roller processes [4, 5].

Formulation of the Problem. A diagram of the flow is shown in Fig. 1. The mass to be processed is fed into the working cavity between the rotor and matrix, is held by those parts and forced through the perforated matrix. In the general case the peripheral velocity of the roller U may not be equal to the translational velocity of the matrix W. We assume that the flow is two-dimensional, laminar, and steady-state. The medium is incompressible. Compared to viscous forces, inertial and mass forces are negligibly small. Commencing from the continuity equation we have $v_x \sim U + W$, $v_y \sim (U + W)h/L$, $L \gg h$, where L and h are the characteristic lengths along the x- and y-axes. Evaluation of the terms of the equations of motion yields $\partial v_x/\partial x \sim (U + W)/L$, $\partial v_x/\partial y \sim (U + W)/h$, $\partial v_y/\partial x \sim (U + W)\cdot h/L^2$. We take $\partial P/\partial y = 0$, i.e., $P = P(x)$. There is no slippage on the working surfaces. The matrix permeability does not depend on its velocity of motion and is characterized by an empirical dependence [1, 2]

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